
Function Representation & Spherical Harmonics

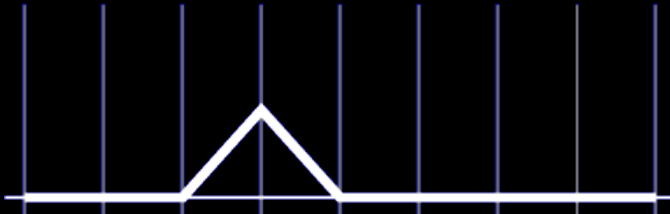
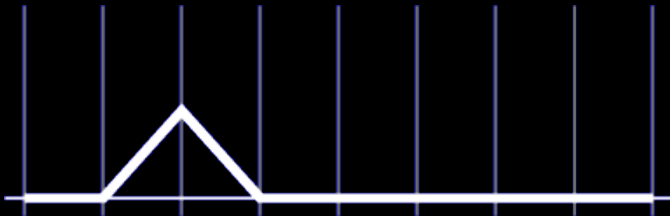
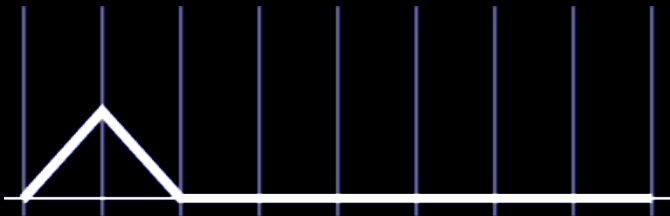
Function approximation

- $G(x)$... function to represent
- $B_1(x), B_2(x), \dots, B_n(x)$... basis functions
- $G(x)$ is a linear combination of basis functions

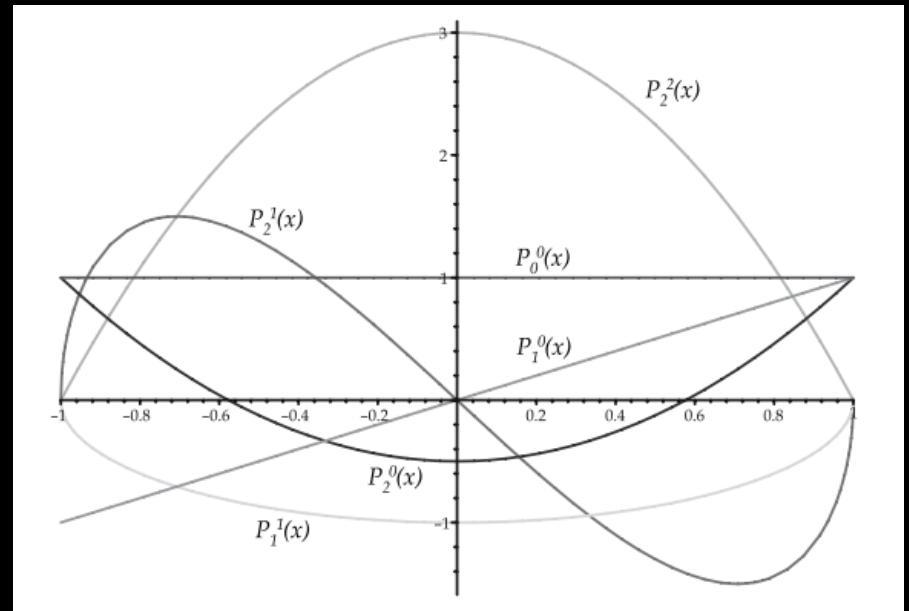
$$G(x) = \sum_{i=1}^n c_i B_i(x)$$

- Storing a finite number of coefficients c_i gives an approximation of $G(x)$

Examples of basis functions



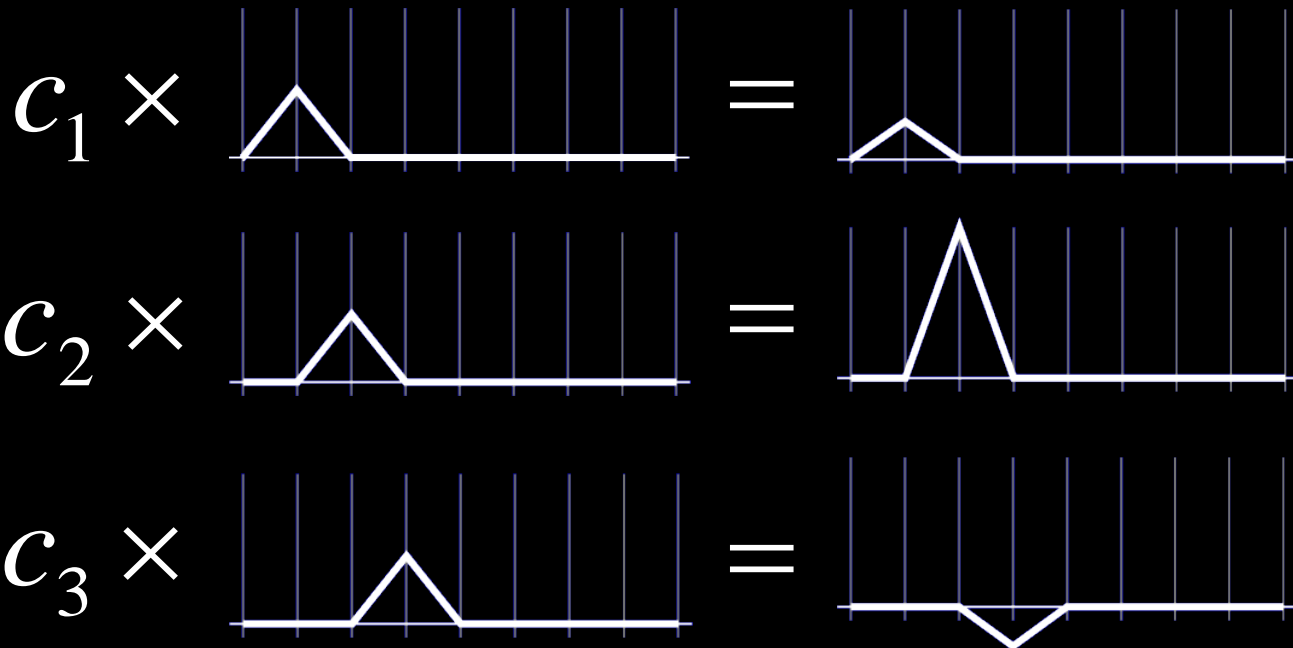
Tent function (linear interpolation)



Associated Legendre polynomials

Function approximation

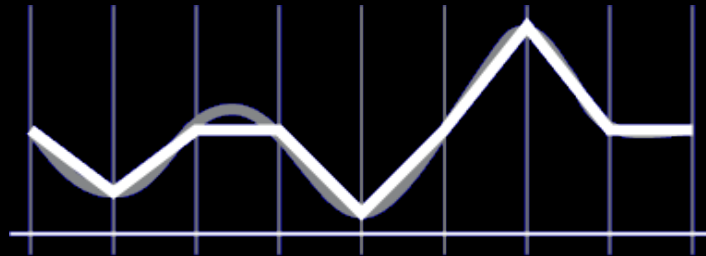
- Linear combination
 - sum of scaled basis functions



Function approximation

- Linear combination
 - sum of scaled basis functions

$$\sum_{i=1}^n c_i B_i(x) =$$



Finding the coefficients

- How to find coefficients c_i ?
 - Minimize an error measure
- What error measure?
 - L_2 error

$$E_{L_2} = \int_I [G(x) - \sum_i c_i B_i(x)]^2$$

Original function

Approximated function

Finding the coefficients

- Minimizing E_{L_2} leads to

$$\begin{bmatrix} \langle B_1 | B_1 \rangle & \langle B_1 | B_2 \rangle & \cdots & \langle B_1 | B_n \rangle \\ \langle B_2 | B_1 \rangle & \langle B_2 | B_2 \rangle & & \vdots \\ \vdots & & \ddots & \vdots \\ \langle B_n | B_1 \rangle & \langle B_n | B_2 \rangle & \cdots & \langle B_n | B_n \rangle \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} \langle G | B_1 \rangle \\ \langle G | B_2 \rangle \\ \vdots \\ \langle G | B_n \rangle \end{bmatrix}$$

Where

$$\langle F | H \rangle = \int_I F(x)H(x)dx$$

Finding the coefficients

- Matrix

$$\mathbf{B} = \begin{bmatrix} \langle B_1 | B_1 \rangle & \langle B_1 | B_2 \rangle & \cdots & \langle B_1 | B_n \rangle \\ \langle B_2 | B_1 \rangle & \langle B_2 | B_2 \rangle & & \vdots \\ \vdots & & \ddots & \vdots \\ \langle B_n | B_1 \rangle & \langle B_n | B_2 \rangle & \cdots & \langle B_n | B_n \rangle \end{bmatrix}$$

does not depend on $G(x)$

- Computed just once for a given basis

Finding the coefficients

- Given a basis $\{B_i(x)\}$
 1. Compute matrix B
 2. Compute its inverse B^{-1}
- Given a function $G(x)$ to approximate
 1. Compute dot products
$$\left[\langle G | B_1 \rangle \quad \langle G | B_2 \rangle \quad \cdots \quad \langle G | B_n \rangle \right]^T$$
 2. ... (next slide)

Finding the coefficients

2. Compute coefficients as

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \mathbf{B}^{-1} \begin{bmatrix} \langle G | B_1 \rangle \\ \langle G | B_2 \rangle \\ \vdots \\ \langle G | B_n \rangle \end{bmatrix}$$

Orthonormal basis

- Orthonormal basis means

$$\langle B_i | B_j \rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

- If basis is orthonormal then

$$\mathbf{B} = \begin{bmatrix} \langle B_1 | B_1 \rangle & \langle B_1 | B_2 \rangle & \cdots & \langle B_1 | B_n \rangle \\ \langle B_2 | B_1 \rangle & \langle B_2 | B_2 \rangle & & \vdots \\ \vdots & & \ddots & \vdots \\ \langle B_n | B_1 \rangle & \langle B_n | B_2 \rangle & \cdots & \langle B_n | B_n \rangle \end{bmatrix} = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix} = \mathbf{I}$$

Orthonormal basis

- If the basis is orthonormal, computation of approximation coefficients simplifies to

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} \langle G | B_1 \rangle \\ \langle G | B_2 \rangle \\ \vdots \\ \langle G | B_n \rangle \end{bmatrix}$$

- We want orthonormal basis functions

Orthonormal basis

- Projection: How “similar” is the given basis function to the function we’re approximating

$$\int \text{[Original function]} \times \text{[Basis function 1]} = c_1$$

$$\int \text{[Original function]} \times \text{[Basis function 2]} = c_2$$

$$\int \text{[Original function]} \times \text{[Basis function 3]} = c_3$$

Original function

Basis functions

Coefficients

Another reason for orthonormal basis functions

- Integral of product = dot product of coefficients

$$f(x) = \begin{array}{|c|} \hline f_i \\ \hline \end{array} \begin{array}{|c|} \hline B_i(x) \\ \hline \end{array}$$

$$g(x) = \begin{array}{|c|} \hline g_i \\ \hline \end{array} \begin{array}{|c|} \hline B_i(x) \\ \hline \end{array}$$

$$\int f(x)g(x)dx = \begin{array}{|c|} \hline f_i \\ \hline \end{array} \begin{array}{|c|} \hline g_i \\ \hline \end{array}$$

Application to GI

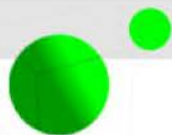
- Reflection equation

$$L_o = \int L_i(\omega_i) \text{BRDF}(\omega_i) \cos \theta_i d\omega_i$$

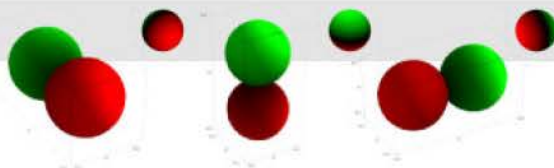
Spherical Harmonics

The real SH basis functions

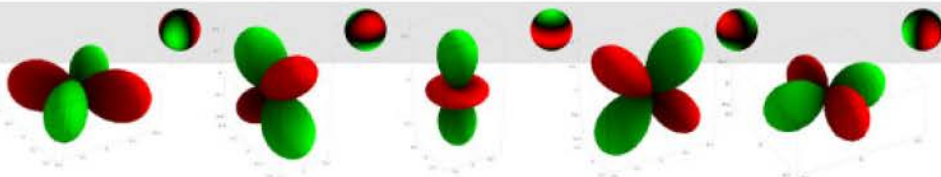
$l=0$



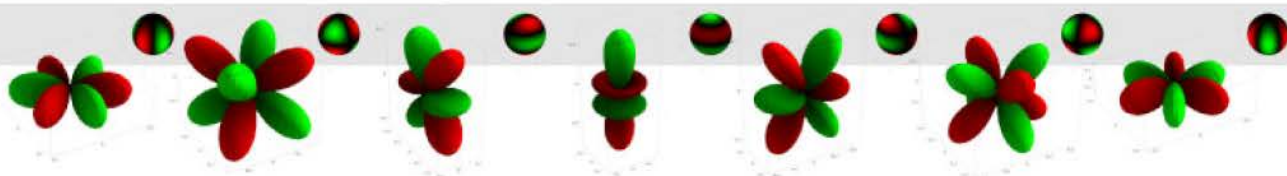
$l=1$



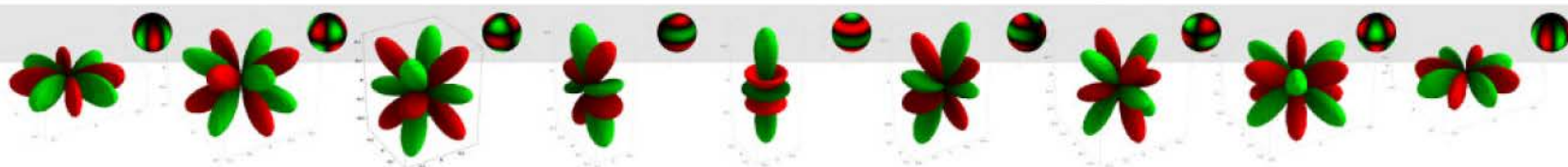
$l=2$



$l=3$



$l=4$



Spherical harmonics

- Spherical function approximation
- Domain $I =$ unit sphere S
 - directions in 3D
- Approximated function: $G(\theta, \varphi)$
- Basis functions: $Y_i(\theta, \varphi) = Y_{l,m}(\theta, \varphi)$
 - indexing: $i = l(l+1) + m$

Spherical harmonics

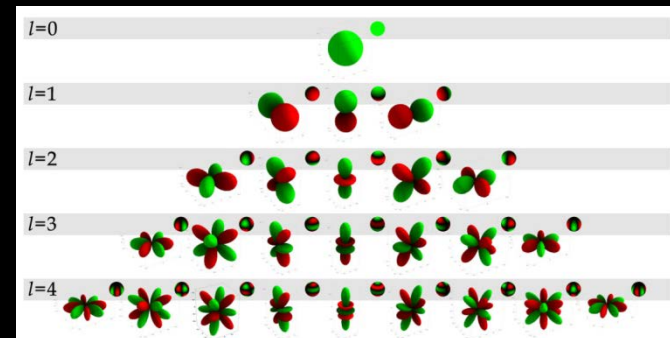
$$y_l^m(\theta, \varphi) = \begin{cases} \sqrt{2} K_l^m \cos(m\varphi) P_l^m(\cos\theta), & m > 0 \\ \sqrt{2} K_l^m \sin(-m\varphi) P_l^{-m}(\cos\theta), & m < 0 \\ K_l^0 P_l^0(\cos\theta), & m = 0 \end{cases}$$

- K ... normalization constant
- P ... Associated Legendre polynomial
 - Orthonormal polynomial basis on $(0,1)$
- In general: $Y_{l,m}(\theta, \varphi) = K \cdot \Psi(\varphi) \cdot P_{l,m}(\cos\theta)$
 - $Y_{l,m}(\theta, \varphi)$ is separable in θ and φ

Function approximation with SH

$$G(\theta, \varphi) = \sum_{l=0}^{n-1} \sum_{m=-l}^{m=l} c_{l,m} Y_{l,m}(\theta, \varphi)$$

- n ...approximation order
- There are n^2 harmonics for order n



Function approximation with SH

- Spherical harmonics are orthonormal

- Function projection

- Computing the SH coefficients:

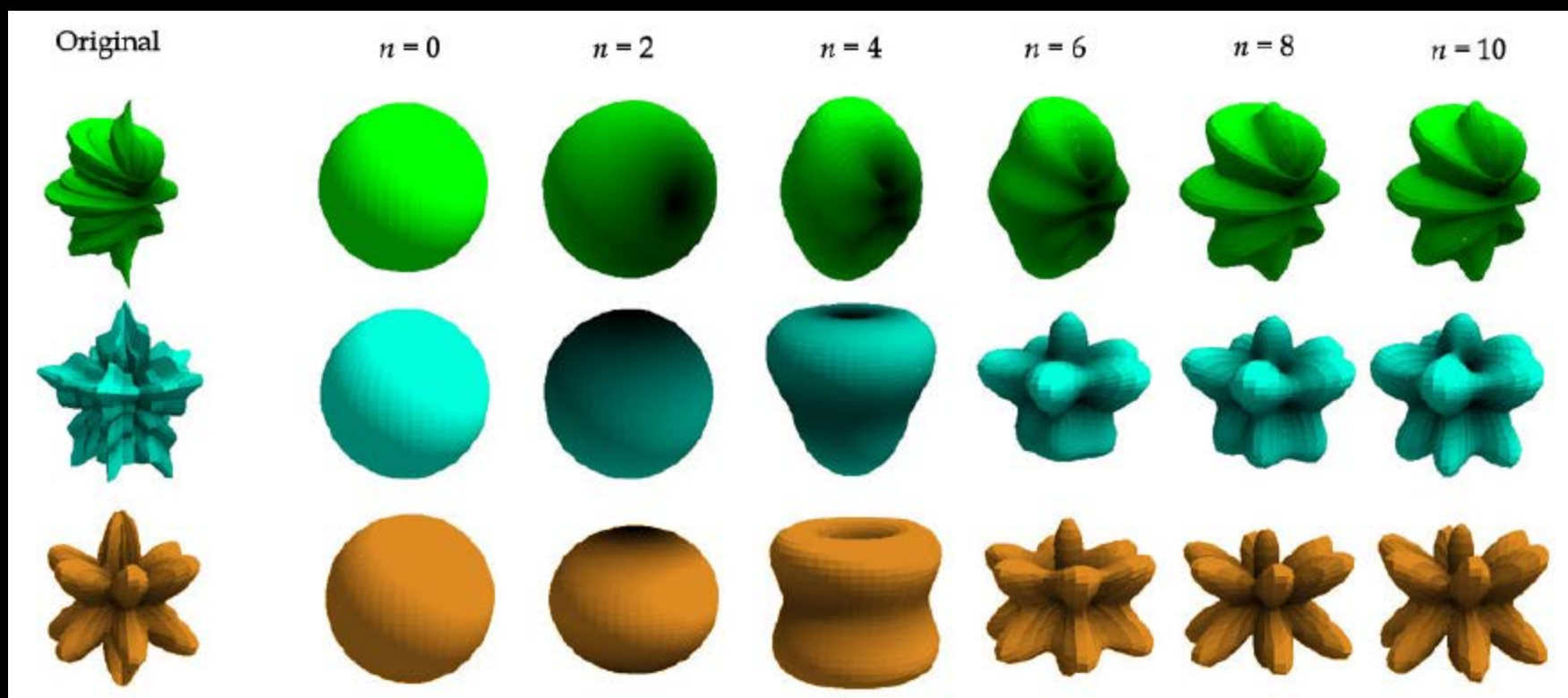
$$\begin{aligned}c_{l,m} &= \langle G | Y_{l,m} \rangle \\ &= \int_S G(\omega) Y_{l,m}(\omega) d\omega \\ &= \int_0^{2\pi} \int_0^\pi G(\theta, \varphi) Y_{l,m}(\theta, \varphi) \sin \theta d\theta d\varphi\end{aligned}$$

- Usually evaluated by numerical integration

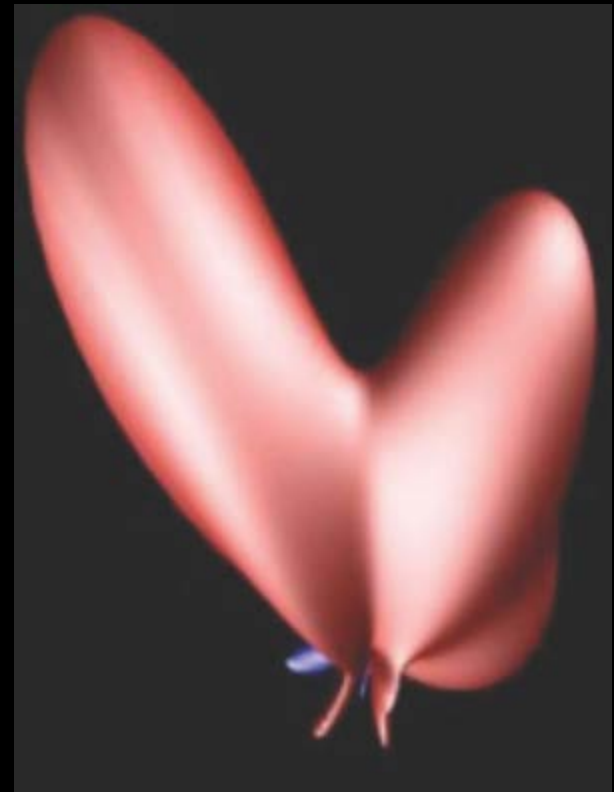
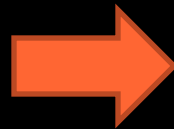
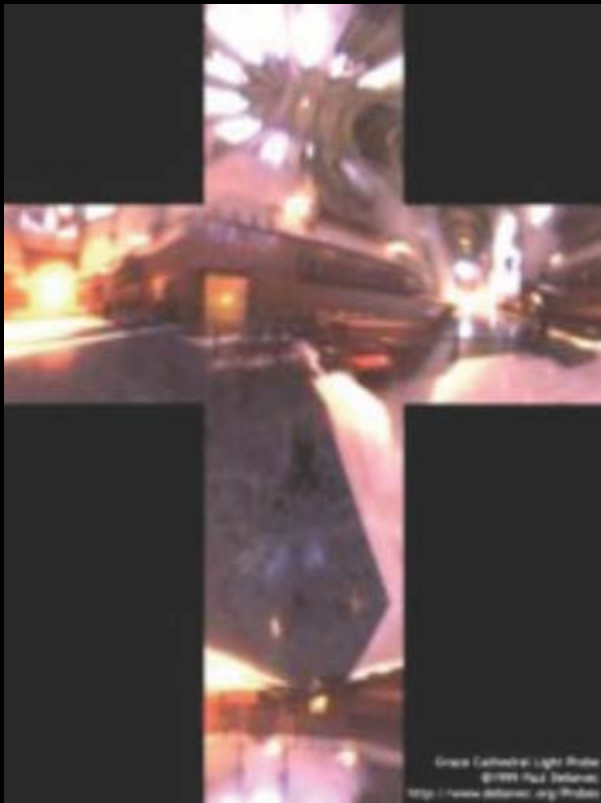
- Low number of coefficients

- low-frequency signal

Function approximation with SH



Environment map approximation with SH

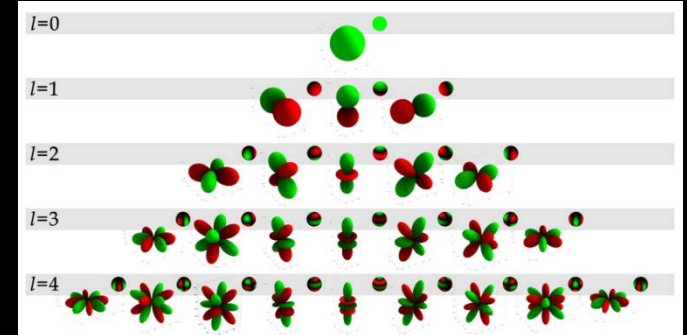


Product integral with SH

- Simplified indexing

- $Y_i = Y_{l,m}$

- $i = l(l+1) + m$



- Two functions represented by SH

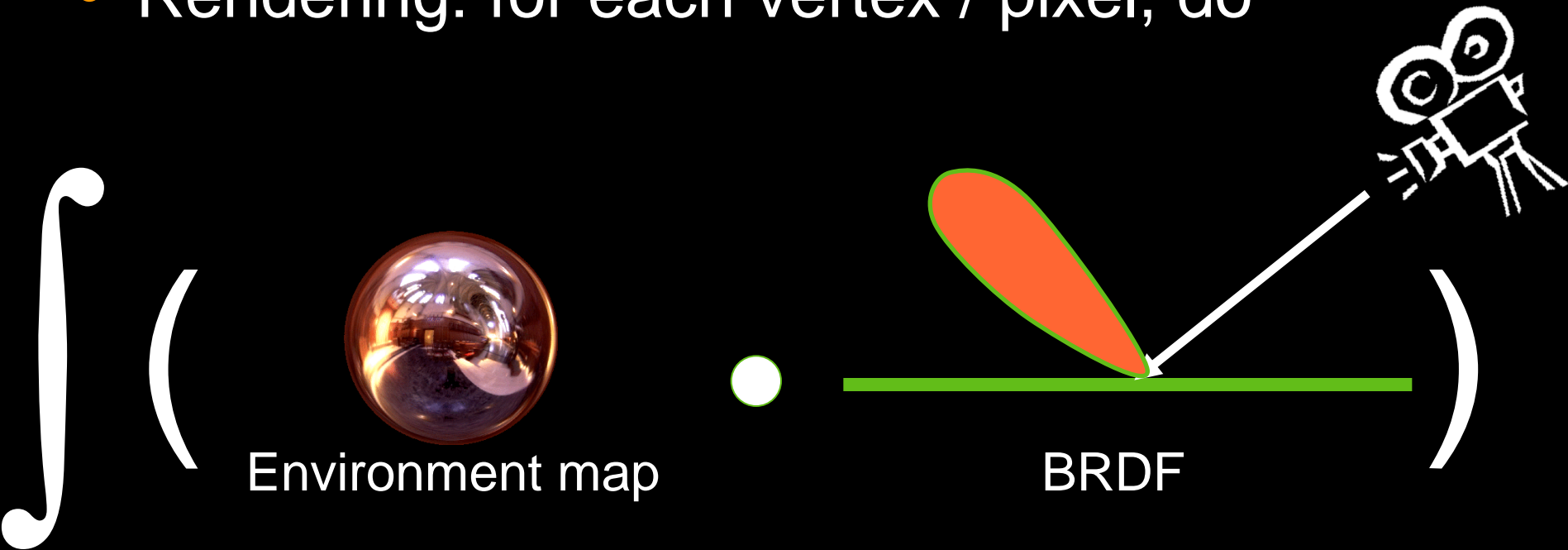
$$F(\omega) = \sum_{i=0}^{n^2} f_i Y_i(\omega)$$

$$G(\omega) = \sum_{i=0}^{n^2} g_i Y_i(\omega)$$

$$\int_S F(\omega) G(\omega) d\omega = \sum_{i=0}^{n^2} f_i g_i$$

Application to reflection equation

- Rendering: for each vertex / pixel, do



= coeff. dot product

$$L_o(\omega_o) = \sum \lambda_i f_i(\omega_o)$$